Second Day. Wednesday October 5th, 2011

4. (Universidad de San Francisco de Quito)

   Let $(b_0, b_1, \ldots, b_{n-1}) = (1, 1, 1, 0, \ldots, 0)$ for $n \geq 3$. Let $C_n = (c_{i,j})_{n\times n}$ be the matrix defined by $c_{i,j} = b_{(j-i) \bmod n}$. Show that $\det(C_n) = 3$ if n is not a multiple of 3 and $\det(C_n) = 0$ if n is a multiple of 3.

   Note: $m \bmod n$ is the remainder of the division of m by n.

5. (Universidad Nacional Autónoma de México) Let n be a positive integer with d nonzero digits. For $k = 0, \ldots, d-1$, define n_k as the number that is obtained by moving the last k digits of n to the front. For example, if $n = 2184$ then $n_0 = 2184$, $n_1 = 4218$, $n_2 = 8421$ and $n_3 = 1842$. For m a positive integer, define $s_m(n)$ as the number of values k such that n_k is a multiple of m. Finally, define a_d as the number of integers n with d nonzero digits for which $s_2(n) + s_3(n) + s_5(n) = 2d$. Find

   $$\lim_{d\to\infty} \frac{a_d}{5^d}\ .$$

6. (Brazilian Mathematics Olympiad) Let Γ be the branch of the hyperbola $x^2 - y^2 = 1$ where $x > 0$. Let $P_0, P_1, \ldots, P_n, \ldots$ be distinct points of Γ with $P_0 = (1, 0)$ and $P_1 = (13/12, 5/12)$. Let t_i be the tangent line to Γ at P_i. Suppose that for all $i \geq 0$ the area of the region delimited by t_i, t_{i+1} and Γ is a constant that does not depend on i. Find the coordinates of the points P_i in terms of i.